

# Enumeration of some classes of words avoiding two generalized patterns of length three\*

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## Abstract

The method we have applied in [BFP] to count pattern avoiding permutations is adapted to words. As an application, we enumerate several classes of words simultaneously avoiding two generalized patterns of length 3.

## 1 Introduction

In the present work we deal with pattern avoidance on words. This topic has first appeared in [R], and has been systematically developed by Burstein in [B]. Subsequently several authors have studied this kind of matters, and in particular in [BM] exact formulas and/or generating functions for the number of words avoiding a single generalized pattern of length 3 have been found. Here we use a general method to count words on a totally ordered alphabet avoiding a set of generalized patterns of length 3 of type  $(1,2)$  (i.e., having a dash between the first and the second element). Our approach consists of inserting a letter at the end of a given word of length  $n$ , thus obtaining a word of length  $n+1$ . We perform this operation in such a way that part of the preceding letters may have to be renamed. The choice of the letter to be inserted depends on the patterns to be excluded. Obviously, the above mentioned insertion technique, if applied to a word avoiding the requested patterns, produces words in which the only occurrence of a forbidden pattern could involve the newly inserted element. Moreover, the particular type of the patterns to be excluded allows to easily control that words of increasing length will be correctly generated. The first appearance of this technique goes back to [BFP]; later, in [E] the author used it to count several classes of generalized pattern avoiding permutations.

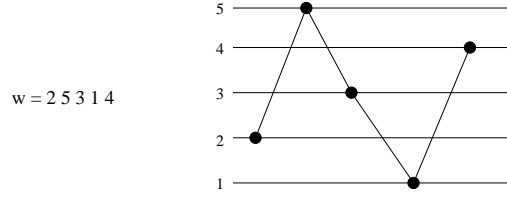
For the sake of clearness, we recall the above mentioned technique in the framework of permutations. Given a permutation  $\pi \in S$  (where  $S$  denotes

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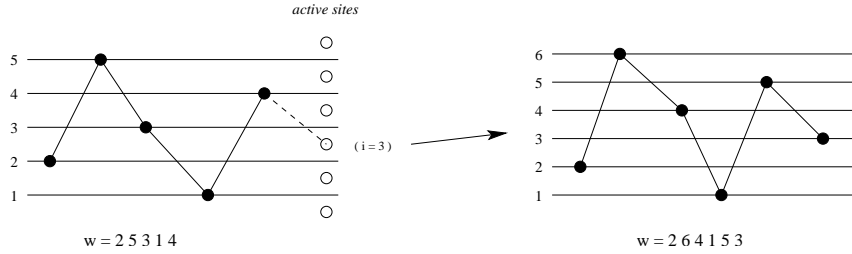
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the whole symmetric group), it can be represented in a path-like form as in the following figure:



Each entry of  $\pi$  is a “node” on the line corresponding to its value. Every pair of adjacent nodes  $\pi_i \pi_{i+1}$  is linked by an ascending or descending segment depending on whether  $\pi_i < \pi_{i+1}$  (i. e., the pair is an “ascent”) or  $\pi_i > \pi_{i+1}$  (i. e., the pair is a “descent”). If  $\pi \in S_n$  (which is the set of all permutations of length  $n$ ), the  $n$  horizontal lines of the permutation divide the plane into  $n + 1$  regions, numbered 1 to  $n + 1$  from bottom to top. Therefore, we can obtain  $n + 1$  permutations belonging to  $S_{n+1}$  starting from  $\pi$ , by inserting a new node in each of these regions (see figure below) and renaming the entries of the new permutation  $\pi' \in S_{n+1}$  according to the following *renaming rule*: if we insert the node into region  $i$ , then

1.  $\pi'_{n+1} = i$
2. for  $j = 1, \dots, n$ 
  - (a) if  $\pi_j < i$  then  $\pi'_j = \pi_j$ ;
  - (b) otherwise  $\pi'_j = \pi_j + 1$ .



Now, we briefly recall the notion of (generalized) pattern avoidance. If  $\pi = \pi_1 \dots \pi_n$  and  $\sigma = \sigma_1 \dots \sigma_k$  are two permutations of  $S_n$  and  $S_k$ , respectively, then  $\pi$  *avoids* the pattern  $\sigma$  if there are no indexes  $i_1 < i_2 < \dots < i_k$  such that  $\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}$  is in the same relative order as  $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k}$ . The subset of  $\sigma$ -avoiding permutations of  $S_n$  is denoted  $S_n(\sigma)$ . A *generalized* pattern  $\tau$  [BS] is a pattern with some dashes inserted, so that two consecutive elements of  $\tau$  are adjacent if there is no dash between them (e.g.  $261-4-35$  is a generalized pattern of length 6). A permutation  $\pi$  *contains* the generalized pattern  $\tau$  if the elements of  $\pi$  corresponding to the elements of  $\tau$  are in the same relative order and any two elements of  $\pi$  corresponding to two adjacent

elements of  $\tau$  are adjacent in  $\pi$  as well. A permutation  $\pi$  *avoids* a generalized pattern  $\tau$  if  $\pi$  does not contain  $\tau$ . For instance, the permutation 41325 contains 321 but not  $32-1$ . In general, if  $T$  is a set of (generalized) patterns,  $S_n(T)$  denotes the set of permutations of  $[n] = \{1, 2, \dots, n\}$  avoiding each pattern of  $T$ .

If we want to generate all the permutations in  $S_{n+1}(T)$  avoiding certain generalized patterns, then the regions where we can insert the new node form a subset of all the  $n + 1$  possible regions, whose elements are called *active sites*. The insertion of the new node to generate a new permutation leads to an important consideration: if  $\pi \in S_n(T)$ , then  $\pi' (\in S_{n+1})$ , obtained from  $\pi$  inserting the last node in some region, does not contain the patterns specified in  $T$  in its entries  $\pi'_j$  with  $j = 1, \dots, n$ , otherwise  $\pi$  itself would contain some pattern of  $T$ . Therefore, we can decide if a region  $i$  is an active site or not simply by checking those generalized patterns the last node is involved in.

For the enumeration of a class of generalized pattern avoiding permutations (according to their length), the following general strategy can be considered:

- find the active sites for the permutations of  $S_n(T)$  among the  $n + 1$  possible regions;
- describe the generation of the permutations of  $S_{n+1}(T)$  and encode the construction with a succession rule;
- obtain the generating function of the sequence enumerating  $|S_n(T)|$  from the succession rule with analytic techniques.

This general technique for the study of the enumerative properties of pattern avoiding permutations can be properly described in the framework of the ECO method [BDLPP], which offers a rigorous setting for the notions of “insertion of a node”, “active site” and “succession rule” mentioned above.

The above described graphical representation for permutations can be easily conformed to words. The only difference lies in the fact that, in the representation of a word, more than one node may appear on the same line.

In our paper, the general strategy we have briefly sketched above is suitably extended for our purpose, adapting it to the case of pattern avoiding words.

## 2 Notations and definitions

Throughout the paper  $M$  will denote a finite totally ordered alphabet, with  $|M| = m$ . Therefore we will always set  $M = \{1, 2, \dots, m\}$ .

A *word pattern* (or simply *pattern*) in  $M$  is a word  $x$  on an alphabet  $\Sigma$ , where  $\Sigma = \{1, \dots, k\}$ , with  $k \leq m$ , where each letter of  $\Sigma$  appears in  $x$  at least once. A word pattern in  $M$  will also be called a *reduced word* on  $M$ . For instance, 32212 is a pattern in  $M = \{1, 2, \dots, 10\}$ , but 41776 isn't.

Denoting as usual by  $M^*$  the set of all the words on the alphabet  $M$ , we say that  $w \in M^*$  *avoids* a pattern  $x$  in  $M$  when no subword of  $w$  is order-isomorphic to  $x$  (see [B]). For example,  $w = 311472511 \in \{1, \dots, 7\}^*$  avoids the patterns 212 and 221, but not the pattern 121.

The notion of generalized pattern for words is analogous to the definition introduced for permutations; however, the rigorous definition can be found in [BM]. In particular, a generalized pattern of length three *of type*  $(1, 2)$  is a reduced word  $x \in \{1, 2, 3\}^*$  of the kind  $a - bc$  (for the meaning of dashes in generalized word patterns, we refer once again to [BM]).

Finally, we will adopt the following notations, which differ from the corresponding ones used in [B, BM]:

- $W_n^{(m)}(T)$ : set of words on  $M$  of length  $n$  avoiding each pattern in the set  $T$ ;
- $f_T^{(m)}(x)$ : generating function of  $W_n^{(m)}(T)$  (with respect to the length of the words).

### 3 The general method

In this section we give a detailed description of the steps to be followed in order to apply our method to the enumeration of classes of (generalized) pattern avoiding words. We remark that all the notations introduced in the present section will be extensively used throughout the paper.

Suppose to be interested in counting the words in  $W_n^{(m)}(T)$  according to their length  $n$ .

Step 1. Denote by  $\widetilde{W}_n^{(k)}(T)$  the set of reduced words of  $W_n^{(m)}(T)$  containing each letter of  $K = \{1, 2, \dots, k\}$ . Our first goal is to determine  $|\widetilde{W}_n^{(k)}(T)|$ , or, at least, an expression for the generating function of such quantities.

Step 2. Now we try to describe an effective construction of the set of the reduced words of length  $n+1$  starting from the set of the reduced words of length  $n$ . To do this, we will apply the so-called ECO method, for which we refer the reader to [BDLPP, DFR, B *et al.*]. Taken a word  $w \in \widetilde{W}_n^{(k)}(T)$ , we try to insert a letter to the right of  $w$ , so obtaining a new word  $w'$  of length  $n+1$ . This can be done by describing  $w$  by means of a path-like representation completely analogous to the one used for permutations. Here, the only difference consists of the fact that we can

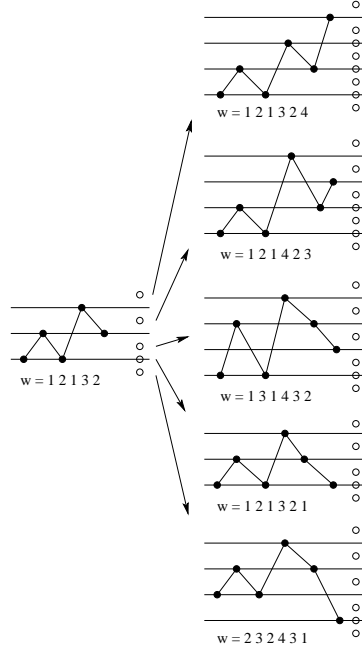


Figure 1: the production of  $w = 12132$

add the rightmost letter both in a region (if the letter occurs for the first time in  $w$ ) and on a horizontal line (otherwise). After the insertion, we possibly need to rename part of the letters of  $w$ , according to the added letter. Moreover, the choice of the letter to be added heavily depends on the patterns to be excluded. As an example, consider  $w = 12132 \in \widetilde{W}_5^{(3)}(1 - 22, 2 - 12)$ . Using the graphical representation of words mentioned in the introduction and denoting with a circle a place where a letter can be added, Figure 1 is obtained.

Figure 1 means that, starting from 12132, we can construct five new words (of length 6), which are the ones described in the picture. As one can immediately notice, if  $w \in \widetilde{W}_n^{(k)}(T)$ , then all the words produced by  $w$  (which, in the sequel, will also be called the *sons* of  $w$ ) belong to  $\widetilde{W}_{n+1}^{(k)}(T) \cup \widetilde{W}_{n+1}^{(k+1)}(T)$ .

Step 3. If we are lucky, the ECO construction found in step 2 can be translated into a succession rule or, which is essentially the same, into a generating tree [W]. In turn, if the succession rule is regular enough, we can hope to use it to get a closed formula for the numbers  $\alpha_{n,k} = |\widetilde{W}_n^{(k)}(T)|$  or maybe, if this is not possible, to derive a nice expression for the generating functions  $f_k(x) = \sum_n \alpha_{n,k} x^n$ , for any fixed  $k$ .

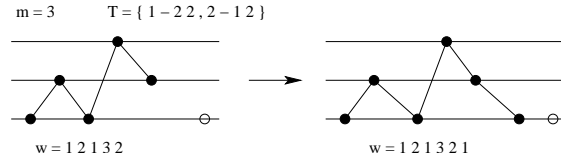
Step 4. If we have been able to enumerate  $\widetilde{W}_n^{(k)}(T)$ , for all  $k \leq m$ , it is now easy to count the words of  $W_n^{(m)}(T)$ . Indeed, it is clear that each reduced word  $w$  in  $\widetilde{W}_n^{(k)}(T)$  is associated with exactly  $\binom{m}{k}$  distinct words in  $W_n^{(m)}(T)$ : just replace the set of letters of  $w$  with any possible subset of  $M$  having  $k$  elements, taking care of preserving the relative order of the letters. For instance, the reduced word  $1213 \in \widetilde{W}_4^{(3)}(1 - 11, 2 - 11)$  is associated with the following  $\binom{5}{3} = 10$  words of  $W_4^{(5)}(1 - 11, 2 - 11)$ :

1213	1214	1215	1314	1315
1415	2324	2325	2425	3435

Thus,  $\binom{m}{k} \alpha_{n,k}$  is the number of words of  $W_n^{(m)}(T)$  having  $k$  distinct letters. Finally, we have simply to sum up to get :

$$|W_n^{(m)}(T)| = \sum_{k=0}^n \binom{m}{k} \alpha_{n,k}.$$

Before applying the above described general method to some specific classes of pattern avoiding words, an important remark is to be done. When we construct the generating tree associated with the ECO construction found in step 2, what we get is not really a description of the growth of  $\widetilde{W}(T)$ . Indeed, if  $|M| = m$ , the construction depicted in Figure 1 only works when the starting word belongs to  $\widetilde{W}_n^{(k)}(T)$ , for  $k < m$ . If  $k = m$ , it is clear that, in our graphical representation, no new horizontal line can be added in order to generate new words. For instance, for  $m = 3$  and  $T = \{1 - 22, 2 - 12\}$ , the only son of the word 12132 is the word 121321 (see figure below).



Therefore, every time we will apply our construction, we have to keep in mind that, for  $k < m$ , it is possible to add new horizontal lines, whereas for  $k \geq m$  no further horizontal line can be added: hence our generation procedure, and the strictly related enumeration technique, must be suitably adapted.

## 4 The class $W(1 - 12, 2 - 21)$

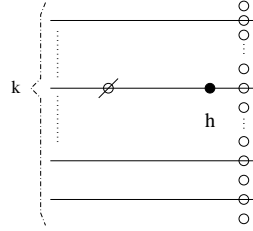
In this section we will apply our method to enumerate the pattern avoiding class  $W_n^{(m)}(1 - 12, 2 - 21)$ . As we will see, this will lead us to consider a

succession rule having only odd labels and, in particular, an infinite number of labels producing only one son.

Following our program, we start by considering the set  $\widetilde{W}_n^{(k)}(1-12, 2-21)$  of the reduced words of  $\widetilde{W}_n^{(m)}(1-12, 2-21)$  on the  $k$ -letter alphabet  $\{1, 2, \dots, k\}$ . Our first goal is to determine  $|\widetilde{W}_n^{(k)}(1-12, 2-21)|$ .

Take  $w \in \widetilde{W}_n^{(k)}(1-12, 2-21)$  and suppose that the last letter of  $w$  is  $h$  ( $\leq k$ ). We can distinguish two cases:

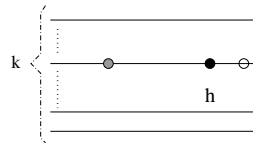
- i) The letter  $h$  occurs in  $w$  for the first time in the last position. We represent this situation using the following diagram:



The above diagram should be interpreted as follows:  $w$  is a reduced word on a  $k$ -letter alphabet; the empty set symbol in the same line of the last letter denote that the  $h$  in the last position is the first occurrence of  $h$  in  $w$ ; finally, the circles on the right indicate the active sites where we can add a new letter in order to obtain a word  $w' \in \widetilde{W}_{n+1}^{(k)}(1-12, 2-21) \cup \widetilde{W}_{n+1}^{(k+1)}(1-12, 2-21)$ . In this particular case, we are allowed to add any letter of  $\{1, 2, \dots, k+1\}$  on the right of  $w$ . To translate this fact into a succession rule, we use the following notations: the label  $(k_h)$  denotes a word of  $\widetilde{W}_n^{(k)}(1-12, 2-21)$  ending with the letter  $h$  and having no further occurrences of  $h$  before; the label  $(\overline{k}_h)$  denotes a word of  $\widetilde{W}_n^{(k)}(1-12, 2-21)$  ending with the letter  $h$  and such that  $h$  also appears in some previous position. With these notations, the production rule encoding the above construction can be written as follows:

$$(k_h) \rightsquigarrow (\overline{k}_1) \cdots (\overline{k}_k)((k+1)_1) \cdots ((k+1)_{k+1}).$$

- ii) The letter  $h$  also occurs in  $w$  in some position other than the last one. In this case, we can use the following diagram:



The meaning of the symbols are obviously the same as for the previous diagram. Here the only difference consists of the fact that  $h$  appears not only in the last position of  $w$ , but also somewhere before, and this has been represented by a gray circle placed in the same line of the last occurrence of  $h$ . In this second case, every insertion in a place (region or line) other than such a line would produce an occurrence of a forbidden pattern. Therefore, with the same notations as above, the associated production rule is the following:

$$(\overline{k}_h) \rightsquigarrow (\overline{k}_h).$$

Now, putting things together, and recalling that the word of minimum length avoiding simultaneously  $1 - 12$  and  $2 - 21$  is  $1$  (which is represented by the label  $(1_1)$ ), the succession rule encoding the whole construction is the following:

$$\left\{ \begin{array}{l} (1_1) \\ (k_h) \rightsquigarrow (\overline{k}_1) \cdots (\overline{k}_k)((k+1)_1) \cdots ((k+1)_{k+1}) \\ (\overline{k}_h) \rightsquigarrow (\overline{k}_h) \end{array} \right. . \quad (1)$$

The above succession rule is indeed too complicated to allow enumeration. However, it is possible to rewrite it in an equivalent form which is surely more suitable for our purposes. To do this, the following two observations are essential.

1. In the production  $(\overline{k}_h) \rightsquigarrow (\overline{k}_h)$ , the subscript  $h$  is unimportant; instead, we should keep track of the parameter  $k$ , since we will use it in the enumeration process. So, since  $(\overline{k}_h)$  has only one production, we can replace this production rule with the rule  $(1_k) \rightsquigarrow (1_k)$  (so that  $(1_k)$  stands for a reduced word on a  $k$ -letter alphabet whose last letter appears somewhere else in the word itself).
2. The production of  $(k_h)$  is independent of  $h$ ; therefore, since  $(k_h)$  produces  $2k+1$  labels, taking care of the previous remark, we can replace its production rule with the following:

$$(2k+1) \rightsquigarrow (1_k)^k(2k+3)^{k+1}.$$

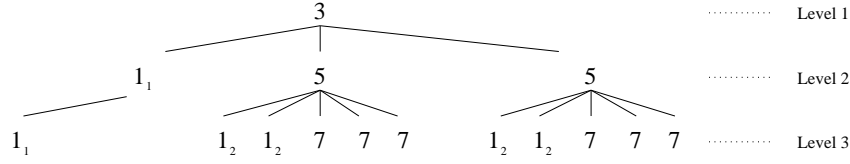
Thus,  $(2k+1)$  represents a  $k$ -letter word on a reduced alphabet whose last letter does not appear in any other position.

Thanks to the above considerations, we have that the rule in (1) is equivalent to:

$$\left\{ \begin{array}{l} (3) \\ (2k+1) \rightsquigarrow (1_k)^k(2k+3)^{k+1} \\ (1_k) \rightsquigarrow (1_k) \end{array} \right. . \quad (2)$$

The first levels of the associated generating tree are depicted in the figure below:





Looking at the labels of the succession rule (2) and keeping in mind how such a rule has been obtained, we immediately observe that:

- i) each label  $(2k + 1)$  represents a word having  $k$  letters;
- ii) each label  $(1_k)$  represents a word having  $k$  letters;
- iii) the unique labels appearing at level  $n$  of the associated generating tree are  $(1_1), (1_2), \dots, (1_{n-1}), (2n + 1)$ .

Therefore,  $\alpha_{n,k}$  (see Section 3) is either the number of labels  $(1_k)$  at level  $n$ , if  $k < n$ , or the number of labels  $(2k + 1)$  at level  $n$ , if  $k = n$ .

From the succession rule (2), we can obtain an explicit formula for the number  $\alpha_{n,k}$ .

**Theorem 4.1** *When  $k \leq m$ , we have*

$$\alpha_{n,k} = \begin{cases} k \cdot k! , & k < n \\ n! , & k = n \end{cases} ,$$

whereas, for  $k > m$ , it is  $\alpha_{n,k} = 0$ .

*Proof.* We can use the rule (2) to get the following recursions for the  $\alpha_{n,k}$ 's:

$$\alpha_{n,n} = n \cdot \alpha_{n-1,n-1} \tag{3}$$

$$\alpha_{n,n-1} = (n - 1) \cdot \alpha_{n-1,n-1} \tag{4}$$

$$\alpha_{n,k} = \alpha_{k+1,k} \quad (k < n - 1). \tag{5}$$

Since  $\alpha_{1,1} = 1$ , from (3) we immediately have  $\alpha_{n,n} = n!$ , whence, from (4), we get  $\alpha_{n,n-1} = (n - 1) \cdot (n - 1)!$ . Finally, from (5), it is  $\alpha_{n,k} = k \cdot k!$ , for  $k < n - 1$ . These formulas hold only for  $k \leq m$ , since our alphabet has  $m$  letters, and so the generating tree of (2) can be used only for  $k < m$ . Obviously, when  $k > m$  we have  $\alpha_{n,k} = 0$ . ■

The first lines of the infinite matrix  $A = (\alpha_{n,k})_{n,k \geq 0}$  are the following (where we have added one more row and one more column to represent the

empty word):

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 6 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 18 & 24 & 0 & \cdots \\ 0 & 1 & 4 & 18 & 96 & 120 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Of course, in the matrix  $A$  we have to turn into 0 the entries of all the columns from the  $(m+1)$ -th onwards.

Now we are ready to get the exact enumeration of the class  $W(1-12, 2-21)$ . Following the method described in section 3, we have the following results.

**Proposition 4.1** *The total number of words of  $W_n^{(m)}(1-12, 2-21)$  having  $k \leq m$  distinct letters is:*

$$\binom{m}{k} \alpha_{n,k} = \begin{cases} k \cdot (m)_k, & k < n \\ (m)_n, & k = n \end{cases},$$

where  $(a)_b$  denotes the usual falling factorial,  $(a)_b = a \cdot (a-1) \cdot \dots \cdot (a-b+1)$ .

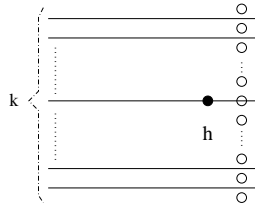
**Theorem 4.2** *It is*

$$|W_n^{(m)}(1-12, 2-21)| = \begin{cases} \sum_{k=0}^{n-1} k \cdot (m)_k + (m)_n, & n \leq m \\ \sum_{k=0}^{m-1} k \cdot (m)_k, & n > m \end{cases}.$$

## 5 The class $W(1-21, 2-12)$

The next case we take into consideration is that of words simultaneously avoiding the two generalized patterns  $1-21$  and  $2-12$ . Also in this case, our technique allows us to get to an explicit counting result quite quickly and easily.

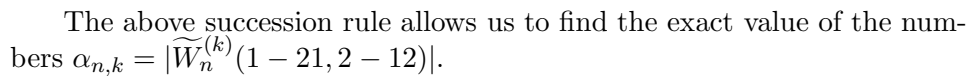
As usual, we start by considering  $\widetilde{W}_n^{(k)}(1-21, 2-12)$ , i.e. reduced words. Suppose that  $w \in \widetilde{W}_n^{(k)}(1-21, 2-12)$  and that  $h$  ( $\leq k$ ) is the last letter of  $w$ . Using a diagram similar to the one of the previous section, we can represent the set of productions of  $w$  as follows:



Unlike the previous class of words, here it is not necessary to distinguish two cases, since the production of  $w$  is independent of the fact that the last letter of  $w$  is the first occurrence of  $h$  or not. Since  $1 \in \widetilde{W}_n^{(k)}(1-21, 2-12)$  is the only word of length 1 of our class, we have the following succession rule associated with the above diagram:

Also in this case, we can observe that the number of labels produced by  $(k_h)$  does not depend on  $h$ . Since we are only interested in the parameter  $k$ , we can easily find that the above rule is equivalent to

In the rule (6), a label  $(k)$  represents a word having  $k-2$  distinct letters. The first few levels of the associated generating tree look as follows:


$$\alpha_{n,k} = k \cdot (n-1)_{k-1}. \quad (7)$$
[illegible]

The level polynomials of the generating tree for arrangements are

$$p_n(x) = \sum_{k \geq 0} (n-1)_k x^k$$

(here the coefficient of  $x^k$ , depending on  $n$ , gives the number of labels  $k+2$  at level  $n$ ). Therefore, the relation between  $p_n(x)$  and the level polynomials  $\alpha_n(x)$  (where the coefficient of  $x^k$  is the number of labels  $k+3$  at level  $n$ ) of our generating tree is expressed by the equality

$$p_n(x) = p_{n-1}(x) + x\alpha_{n-1}(x),$$

which gives

$$\alpha_n(x) = \frac{p_{n+1}(x) - p_n(x)}{x},$$

whence, recalling that the coefficients of the polynomials  $\alpha_n(x)$  coincide with the numbers  $\alpha_{n,k}$  only when  $k \leq m$ , for such values of  $k$  we have

$$\alpha_{n,k} = (n)_k - (n-1)_k = k \cdot (n-1)_{k-1}. \quad \blacksquare$$

The first few lines of the matrix  $A = (\alpha_{n,k})_{n,k \geq 0}$  (where the first column, representing the set of words having 0 letters, and the first row have been added) are:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 6 & 0 & 0 & \cdots \\ 0 & 1 & 6 & 18 & 24 & 0 & \cdots \\ 0 & 1 & 8 & 36 & 96 & 120 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As usual, we must remember that the above formula is only valid when  $k \leq m$ , so that in the matrix  $A$  we have to consider columns exclusively up to  $k = m$ , setting all the remaining entries equal to 0.

Finally we can give complete enumeration results.

**Proposition 5.2** *The number of words of  $W_n^{(m)}(1-21, 2-12)$  having  $k$  distinct letters is*

$$\binom{m}{k} \alpha_{n,k} = \binom{m}{k} k \cdot (n-1)_{k-1}.$$

**Theorem 5.1** *The total number of words of  $W_n^{(m)}(1-21, 2-12)$  is*

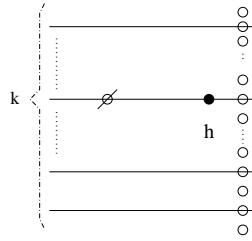
$$|\widetilde{W}_n^{(k)}(1-21, 2-12)| = \sum_{k=0}^n \binom{m}{k} k \cdot (n-1)_{k-1}.$$

## 6 The class $W(1-11, 1-12)$

This case is more difficult to deal with than the previous ones. Of course, we will follow the same general method; at the end, we will get an expression for the generating function of the class, rather than a closed form for the associated numerical sequence.

As usual, we start by considering the set  $\widetilde{W}_n^{(k)}(1-11, 1-12)$  of  $k$ -reduced words of length  $n$  simultaneously avoiding  $1-11$  and  $1-12$ . Let  $w \in \widetilde{W}_n^{(k)}(1-11, 1-12)$  with last letter  $h$ . We can distinguish two cases.

- i) The occurrence of  $h$  in the last position of  $w$  is the first occurrence of  $h$  in  $w$ . Drawing the usual diagram, we are in the following situation:

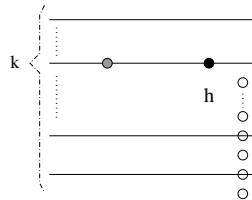


Remember that the circles on the right represent those active sites in which we are allowed to add a letter at the end of  $w$ . In this case, the above diagram tells us that we can add a letter in any position. Thus we can encode this fact using the following production rule:

$$(k_h) \rightsquigarrow (\overline{k}_1) \cdots (\overline{k}_k)((k+1)_1) \cdots ((k+1)_{k+1}) \quad (8)$$

(the meaning of the symbols has been extensively explained in the previous two cases).

- ii) The letter  $h$  occurs in  $w$  for the first time somewhere before its occurrence in the last position. In this case the situation is the following:



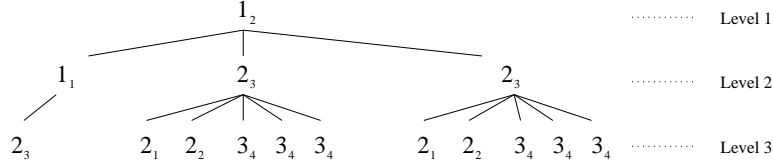
Here the production is not as trivial as in case i), and can be encoded by the rule

$$(\overline{k}_h) \rightsquigarrow (\overline{k}_1) \cdots (\overline{k}_{h-1})((k+1)_1) \cdots ((k+1)_h). \quad (9)$$

What we can immediately observe is that, in case i), the production of  $(k_h)$  does not depend on  $h$ , whereas, in case ii), the production of  $(\bar{k}_h)$  does. Moreover, setting  $h = k + 1$  both in (8) and in (9), we obtain formally the same production rule. This implies that our ECO construction for the class  $\widetilde{W}_n^{(k)}(1 - 11, 1 - 12)$  is equivalent to the following:

$$\begin{cases} (1_2) \\ (k_h) \rightsquigarrow (k_1) \cdots (k_{h-1})((k+1)_{k+2})^h \end{cases}, \quad h \leq k+1 \quad , \quad (10)$$

where the label  $(k_h)$ , for  $h < k + 1$ , corresponds to the labels  $(\bar{k}_h)$  in (8) and (9) and the label  $(k_{k+1})$  corresponds to the label  $(k_h)$  in (8) and (9), *for any*  $h \leq k + 1$ . We can also have a look at the first levels of the associated generating tree:



Using the succession rule (10), we are able to enumerate the class  $\widetilde{W}_n^{(k)}(1 - 11, 1 - 12)$ . To do this, we will make use of a recursion on the entries of the associated ECO matrix we are going to show below.

The first lines of the ECO matrix associated with the above depicted generating tree (i.e., the infinite matrix describing the distribution of the labels at the various levels of the generating tree, see [DFR]) can be organized as follows:

labels→ at level ↓	$1_2$	$1_1$	$2_3$	$2_2$	$2_1$	$3_4$	$3_3$	$3_2$	$3_1$	$4_5$	$4_4$	$4_3$	$4_2$	$4_1$	$5_6$	...
1	1	0														
2	0	1														
3	0	0	2	0	0											
4	0	0	1	2	2	6	0	0	0							
5	0	0	0	1	3	9	6	6	6	24	0	0	0	0		
			0	0	1	5	9	15	21	72	24	24	24	24	120	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

As it should be clear, since  $(k_h)$  represents a word having  $k$  distinct letters, the cardinality of the set of words having  $k$  distinct letters is given by the sum of the entries of the columns labelled  $k_h$ , for  $h$  running from 1 to  $k + 1$ . To get the desired recurrence relation among the entries of the matrix, we recall that the columns labelled  $k_h$  ( $1 \leq h \leq k$ ) derive from the production rule (9), whereas those labelled  $k_{k+1}$  derive from the production

rule (8). In the sequel we will denote by  $C_k(x)$  the generating function of column  $k_{k+1}$  and by  $C_k^{(h)}(x)$  the generating function of column  $k_h$ , with  $h \neq k+1$ . Columns labelled  $k_{k+1}$  will also be called *special columns*. To keep track of the empty word we will also consider the generating function  $C_0(x) = 1$ .

**Proposition 6.1** *The following recurrence relation holds:*

$$\begin{aligned} C_0(x) &= 1 \\ C_k(x) &= \sum_{j=1}^k \binom{k}{j} x^j C_{k-1}(x), \quad k \geq 1. \end{aligned}$$

*Proof.* Looking at the succession rule (10), we observe that a label  $(k_h)$ , for  $1 \leq h \leq k$ , can only be generated by a label  $(k_t)$ , with  $t > h$ , and each of these labels produces  $(k_h)$  precisely once. Moreover, a label  $(k_{k+1})$  can be generated by any label  $((k-1)_h)$ , and  $((k-1)_h)$  produces precisely  $h$  copies of  $(k_{k+1})$ . Starting from these considerations, we are led to the following set of recursions for the entries of our ECO matrix:

$$\begin{aligned} a(1, 1_2) &= 1, & a(n, 1_2) &= 0 \quad (n > 1), \\ a(n, k_h) &= \sum_{i=h+1}^{k+1} a(n-1, k_i) \quad (h \leq k), \\ a(n, k_{k+1}) &= \sum_{i=1}^k a(n-1, (k-1)_i) + \sum_{i=1}^{k-1} a(n, (k-1)_i), \end{aligned}$$

where, of course,  $a(n, k_h)$  denotes the entry corresponding to row  $n$  and column labelled  $k_h$ .

The recursion for  $a(n, k_h)$ , with  $h \leq k$ , can be iterated so to obtain a formula expressing  $a(n, k_h)$  only in terms of the entries of special columns, which is:

$$a(n, k_h) = \sum_{j=0}^{k-h} \binom{k-h}{j} a(n-1-j, k_{k+1}). \quad (11)$$

The above formula can be proved using an easy induction argument on  $h$ .

Now we can plug the above expression for  $a(n, k_h)$  into the recursion formula for  $a(n, k_{k+1})$ , thus obtaining:

$$\begin{aligned} a(n, k_{k+1}) &= a(n-1, (k-1)_k) \\ &+ \sum_{i=1}^{k-1} \sum_{j=0}^{k-1-i} \binom{k-1-i}{j} (a(n-2-j, (k-1)_k) + a(n-1-j, (k-1)_k)). \end{aligned}$$

The above recursion for the entries of special columns can be immediately translated into the desired recursion for the generating function  $C_k(x)$ . ■

**Corollary 6.1**

$$C_k(x) = \sum_{i=k}^{\frac{k^2+k}{2}} \left( \sum_{\substack{j_1, \dots, j_k > 0 \\ j_1 + \dots + j_k = i}} \binom{k}{j_k} \binom{k-1}{j_{k-1}} \cdots \binom{1}{j_1} \right) x^i. \quad (12)$$

Formula (12) is a completely explicit expression for the generating function  $C_k(x)$ , but it is of course impossible to use it in any expression in which  $C_k(x)$  is required. For this reason, in the sequel we will simply write  $C_k(x)$  in our formulas, but the reader should remember that  $C_k(x)$  can be replaced by its expression in formula (12).

In order to completely know the entries of our ECO matrix, we now need to find an expression for the generating functions  $C_k^{(h)}(x)$ .

**Proposition 6.2** *The following formula (expressing  $C_k^{(h)}(x)$  in terms of  $C_k(x)$ ) holds:*

$$C_k^{(h)}(x) = x(1+x)^{k-h} C_k(x).$$

*Proof.* Converting into generating functions the recurrence relation (11) immediately gives the required formula. ■

Now we are ready to reach our first goal, that is the enumeration of the class of words  $\widetilde{W}_n^{(k)}(1-11, 1-12)$ .

**Theorem 6.1** *The generating function  $f_k(x)$  of  $k$ -reduced words of  $W_n^{(m)}(1-11, 1-12)$  according to the length is*

$$f_k(x) = (1+x)^k C_k(x).$$

*Proof.* Since  $k$ -reduced words of  $W_n^{(m)}(1-11, 1-12)$  are represented by the labels of the type  $(k_h)$ , with  $1 \leq h \leq k+1$ , the generating function  $f_k(x)$  can be obtained by simply summing up the generating functions corresponding to such labels, whence

$$\begin{aligned} f_k(x) &= C_k(x) + \sum_{h=1}^k C_k^{(h)}(x) \\ &= C_k(x) + \sum_{h=1}^k x(1+x)^{k-h} C_k(x) \\ &= C_k(x) \cdot \left( 1 + x \cdot \sum_{h=0}^{k-1} (1+x)^h \right) \\ &= (1+x)^k C_k(x). \quad \blacksquare \end{aligned}$$



Before performing the last step of our methodology, the usual remark is in order. It is obvious that the above result is valid only when  $k$  is “small”. More precisely, the reader has to remember that, if the starting alphabet has  $m$  letters, then in the above ECO matrix we must consider only the labels  $(k_h)$  with  $k \leq m$ .

The last two theorems, stated as usual without proofs, conclude our enumeration of  $W_n^{(m)}(1-11, 1-12)$ .

**Theorem 6.2** *The generating function for the number of words of  $W_n^{(m)}(1-11, 1-12)$  having  $k$  distinct letters is*

$$\binom{m}{k} (1+x)^k C_k(x).$$

**Theorem 6.3** *The generating function of the sequence  $|W_n^{(m)}(1-11, 1-12)|$  is*

$$f_{\{1-11, 1-12\}}^{(m)} = \sum_{k=0}^m \binom{m}{k} (1+x)^k C_k(x).$$

## 7 Further work

In our paper we have given complete results concerning the enumeration of the classes of words  $W(T)$ , where  $T$  consists of two generalized patterns of length three of type  $a-bc$  on the alphabet  $\{1, 2\}$ . Our main aim was to illustrate the soundness of our methodology, as well as to use it to find some new counting results. Of course, to complete the enumeration of  $W(T)$  when  $T$  is as above, many more cases should be considered. Following essentially the same lines, it is possible to determine some further generating functions, which we express below without proof.

**Theorem 7.1** *The generating function of  $|W_n^{(m)}(1-11, 1-21)|$  is*

$$f_{\{1-11, 1-21\}}^{(m)} = \sum_{k=0}^m \binom{m}{k} x(1+x)^k \left( \prod_{i=0}^{k-2} ((1+x)^{k-i} - 1) \right)$$

**Theorem 7.2** *The generating function of  $|W_n^{(m)}(1-11, 1-22)|$  is*

$$f_{\{1-11, 1-22\}}^{(m)} = \sum_{k=0}^m \binom{m}{k} \frac{x^k (x+k)_k}{\prod_{i=1}^{k-1} (1-ix)}.$$

**Theorem 7.3** *The generating function of  $|W_n^{(m)}(2-11, 1-22)|$  is*

$$f_{\{2-11, 1-22\}}^{(m)} = \sum_{k=0}^m \frac{(m)_k \cdot x^{k-1}}{\left( \prod_{i=1}^{k-1} (1-ix) \right) \cdot (1-x)}.$$

The remaining cases seem to be more difficult to be dealt with; up to Wilf-equivalence, they are the following:

$$\begin{array}{ll}
W_n^{(m)}(1-12, 1-21) & W_n^{(m)}(1-12, 1-22) \\
W_n^{(m)}(1-12, 2-11) & W_n^{(m)}(1-12, 2-12) \\
W_n^{(m)}(1-21, 1-22) & W_n^{(m)}(1-21, 2-11).
\end{array}$$

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